Confinement of fermions by mixed vector-scalar linear potentials in two-dimensional space-time

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Abstract

The problem of confinement of fermions in 1+1 dimensions is approached with a linear potential in the Dirac equation by considering a mixing of Lorentz vector and scalar couplings. Analytical bound-states solutions are obtained when the scalar coupling is of sufficient intensity compared to the vector coupling.

The Coulomb potential of a point electric charge in a 1+1 dimension, considered as the time component of a Lorentz vector, is linear and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [1]-[2] and in the Thirring-Schwinger model [3]. It is frustrating that, due to the tunneling effect (Klein´s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [4]-[5]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [6]-[7]. Recently it was incorrectly concluded that even in this case there are no bound states [8], despite the absence of Klein´s paradox. Later, the proper solutions for this last problem were found [9]-[11]. However, it is well known from the quarkonium phenomenology in the real 3+1 dimensional world that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, *e.g.*, [12]). Therefore, the problem of confinement of fermions by a linear potential in 1+1 dimensions deserves a more general analyses. With this in mind, we approach in the present paper the Dirac equation in one-plus-one dimensions with a linear potential considering it as a more general mixing of Lorentz vector and scalar couplings. It is found that there are analytical bound-state solutions on condition that the scalar component of the potential is of sufficient strength compared to the vector component ($|V_s| \geq |V_t|$). As a by-product, the present approach also provides the opportunity to find that there exist relativistic confining potentials providing no bound-state solutions in the nonrelativistic limit. Although we shall confine our discussion to the vector-scalar mixing, the inclusion of a pseudoscalar potential can be allowed. The hearth of the matter is a unitary transformation similar to that one recently applied to the vector-pseudoscalar mixing in 3+1 dimensions [13]. That unitary transformation leads to a Sturm-Liouville eigenvalue problem for the upper component of the Dirac spinor.

Let us begin by presenting the Dirac equation in 1+1 dimensions. In the presence of a time-independent potential the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass m reads

$$\mathcal{H}\Psi = E\Psi \tag{1}$$

$$\mathcal{H} = c\alpha p + \beta mc^2 + \mathcal{V} \tag{2}$$

where E is the energy of the fermion, c is the velocity of light and p is the momentum operator. α and β are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. From the last two relations it steams that both α and β are traceless and have eigenvalues equal to -1, so that one can conclude that α and β are even-dimensional matrices. One can choose the 2×2 Pauli matrices satisfying the same algebra as α and β, resulting in a 2-component spinor Ψ. The positive definite function $|\Psi|^2 = \Psi^\dagger \Psi$, satisfying a continuity equation, is interpreted as a probability position density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states [14]. We use $\alpha = \sigma_1$ and $\beta = \sigma_3$. For the potential matrix we consider

$$\mathcal{V} = 1V_t + \beta V_s + \alpha V_e + \beta\gamma^5 V_p \tag{3}$$

where 1 stands for the 2×2 identity matrix and $\beta\gamma^5 = \sigma_2$. This is the most general combination of Lorentz structures for the potential matrix because there are only four linearly independent 2×2 matrices. The subscripts for the terms of potential denote their properties under a Lorentz transformation: t and e for the time and space components of the 2-vector potential, s and p for the scalar and pseudoscalar terms, respectively.

Defining the spinor ψ as

$$\psi = \exp\left(\frac{i}{\hbar}\Lambda\right)\Psi \tag{4}$$

where

$$\Lambda(x) = \int^x dx' \frac{V_e(x')}{c} \tag{5}$$

the space component of the vector potential is gauged away

$$\left(p + \frac{V_e}{c}\right)\Psi = \exp\left(\frac{i}{\hbar}\Lambda\right)p\psi \tag{6}$$

so that the time-independent Dirac equation can be rewritten as follows:

$$H\psi = E\psi \tag{7}$$

$$H = \sigma_1 cp + \sigma_2 V_p + \sigma_3\left(mc^2 + V_s\right) + 1V_t \tag{8}$$

showing that the space component of a vector potential only contributes to change the spinors by a local phase factor. Introducing the unitary operator

$$U(\theta) = \exp\left(-i\frac{\theta}{2}\sigma_1\right) \tag{9}$$

where θ is a real quantity such that $-\pi \leq \theta \leq \pi$, the transform of the Hamiltonian (8) takes the form

$$\begin{aligned} \widetilde{H} &= UHU^{-1} = \sigma_1 cp + \sigma_2 \left[V_p \cos(\theta) - \left(mc^2 + V_s\right)\sin(\theta)\right] \\ &\quad +\sigma_3 \left[\left(mc^2 + V_s\right)\cos(\theta) - V_p \sin(\theta)\right] + V_t \end{aligned} \tag{10}$$

In terms of the upper and the lower components of the transform of the spinor ψ under the action of the operator U:

$$\widetilde{\psi} = U\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{11}$$

and, moreover, choosing

$$V_t = V_s \cos(\theta) - V_p \sin(\theta) \tag{12}$$

the Dirac equation decomposes into:

$$\begin{aligned} &-\hbar^2 c^2 \phi'' + \Big\{\hbar c \left[V_s' \sin(\theta) + V_p' \cos(\theta)\right] + \left[\left(mc^2 + V_s\right)\sin(\theta) + V_p \cos(\theta)\right]^2 \\ &+2\left[E + mc^2\cos(\theta)\right]\left[V_s \cos(\theta) - V_p \sin(\theta)\right] - \left[E^2 - m^2c^4\cos^2(\theta)\right]\Big\}\phi = 0 \end{aligned} \tag{13}$$

$$\chi = i\frac{-\hbar c\phi' + \left[\left(mc^2 + V_s\right)\sin(\theta) + V_p\cos(\theta)\right]\phi}{E + mc^2\cos(\theta)} \tag{14}$$

where the prime denotes differentiation with respect to x. In terms of the upper and the lower components the spinor is normalized as

$$\int_{-\infty}^{+\infty}\left(|\phi|^2 + |\chi|^2\right)dx = 1 \tag{15}$$

so that both ϕ and χ are square integrable functions.

Now, we shall restrict our discussion to linear potentials with $V_p = 0$, namely

$$\begin{aligned} V_s &= a_s|x| \\ V_t &= V_s \cos(\theta) \end{aligned} \tag{16}$$

i.e., $|V_s| \geq |V_t|$. Thus, Eq.(13) becomes the Schrödinger-like equation

$$\begin{aligned} -\hbar^2 c^2 \phi'' + \Big\{ a_s^2 \sin^2(\theta) x^2 + 2a_s \left[E\cos(\theta) + mc^2 \right] |x| \\ + \varepsilon(x) \hbar c a_s \sin(\theta) - \left(E^2 - m^2 c^4 \right) \Big\} \phi = 0 \end{aligned} \tag{17}$$

where $\varepsilon(x) = x/|x|$, for $x \neq 0$. It is clear that Eq.(17) allows two distinct classes of solutions depending on $\sin(\theta)$.

For the class $\sin(\theta) = 0$, we define

$$\zeta_\pm = \alpha_\pm |x| + \beta_\pm \tag{18}$$

where the plus sign corresponds to $V_t = V_s$ ($\cos(\theta) = 1$) and the minus sign corresponds to $V_t = -V_s$ ($\cos(\theta) = -1$). Furthermore,

$$\begin{aligned} \alpha_\pm &= \pm \left[\frac{2a_s}{\hbar^2 c^2} \left(E \pm mc^2 \right) \right]^{1/3} \\ \beta_\pm &= \mp \frac{\alpha_\pm}{2a_s} \left(E \mp mc^2 \right) \end{aligned} \tag{19}$$

so that Eq.(17) turns into the Airy differential equation

$$\frac{d^2\phi(\zeta_\pm)}{d\zeta_\pm^2} - \zeta_\pm \phi(\zeta_\pm) = 0 \tag{20}$$

which has square integrable solutions expressed in terms of the Airy functions [15]: $\phi(\zeta_\pm) = A_\pm \, Ai(\zeta_\pm)$, where $A_\pm$ is a normalization constant. The joining condition of ϕ and its derivative at $x = 0$ leads to the quantization conditions

$$
\begin{aligned}
Ai(\beta_\pm) &= 0 \quad \text{for odd parity solutions} \\
Ai'(\beta_\pm) &= 0 \quad \text{for even parity solutions}
\end{aligned} \tag{21}
$$

These quantization conditions have solutions only for $\beta_\pm < 0$ and some of them is listed are Table I [15]. One can see that $\beta_\pm < 0$ always corresponds to $|E| > mc^2$. Substitution of the roots of $Ai(\beta_\pm)$ and $Ai'(\beta_\pm)$ into (19) allow us to obtain the possible energies as the solutions of a forth-degree algebraic equation:

$$
E^4 \mp 2mc^2E^3 \pm 2m^3c^6E - \left[m^4c^8 + (2\hbar c a_s)^2 |\beta_\pm|^3\right] = 0 \tag{22}
$$

Instead of giving explicit solutions to this algebraic equation in terms of radicals, we satisfy ourselves verifying that the roots of Eq. (22) always satisfy the requirement $|E| > mc^2$ by using the Descartes´ rule of signs (henceforth DRS). The DRS states that an algebraic equation with real coefficients $a_k\lambda^k + ... + a_1\lambda + a_0 = 0$ the difference between the number of changes of signs in the sequence $a_k, ..., a_1, a_0$ and the number of positive real roots is an even number or zero, with a root of multiplicity k counted as k roots and not counting the null coefficients (see, *e.g.*, [17]-[18]). The verification of the existence of solutions for $E > mc^2$ is made simpler if we write $E = mc^2 + \delta$. We get

$$
\begin{aligned}
\delta^4 + 2mc^2\delta^3 - (2\hbar c a_s)^2 |\beta_+|^3 &= 0 \\
\delta^4 + 6mc^2\delta^3 + 12m^2c^4\delta^2 + 8m^3c^6\delta &- (2\hbar c a_s)^2 |\beta_-|^3 = 0
\end{aligned} \tag{23}
$$

Observing the difference of signs among the coefficients of the leading coefficient and the lowest degree it becomes clear that there exist positive roots. In this particular case the DRS assures that there exists just one solution, since there is only one change of sign in the sequence of coefficients of (23). It is interesting to note that this result is true whatever the fermion masses and the coupling constant. The very same conclusion for $E < mc^2$ can be obtained by observing that the upper-sign solutions ($V_t = V_s$) are mapped into the lower-sign solutions ($V_t = -V_s$), and vice-versa, by the change $E \to -E$.

On the other hand, for the class $\sin(\theta) \neq 0$, corresponding to $|V_t| < |V_s|$, we define

$$\xi = \frac{\xi_0}{x_0}\left(|x| + x_0\right)$$

where

$$\begin{aligned} x_0 &= \frac{E\cos(\theta) + mc^2}{a_s \sin^2(\theta)} \\ \xi_0 &= \sqrt{\frac{2|a_s \sin(\theta)|}{\hbar c}} x_0 \end{aligned} \tag{24}$$

Moreover,

$$\nu = -1 + \frac{\left[E + mc^2\cos(\theta)\right]^2 - m^2c^4\cos(2\theta)}{2\hbar c a_s \sin^3(\theta)} \tag{25}$$

such that

$$-\frac{d^2\phi(\xi)}{d\xi^2} + \frac{\xi^2}{4}\phi(\xi) = \begin{cases} (\nu + 1/2)\,\phi(\xi) & x > 0 \\ (\nu + 3/2)\,\phi(\xi) & x < 0 \end{cases} \tag{26}$$

whose solutions are the square integrable parabolic cylinder functions [15]: $\phi(\xi) = BD_\nu(\xi)$, for $x > 0$, and $\phi(\xi) = CD_{\nu+1}(\xi)$, for $x < 0$. B and C are normalization constants. Making use of the recurrence formulas

$$\begin{aligned} D'_\nu(z) - \frac{z}{2}D_\nu(z) + D_{\nu+1}(z) &= 0 \\ D'_\nu(z) + \frac{z}{2}D_\nu(z) - \nu D_{\nu-1}(z) &= 0 \end{aligned} \tag{27}$$

and the matching conditions at $x = 0$, the quantization condition is

$$D_{\nu+1}\left(\xi_0\right) = \pm\sqrt{\nu + 1}D_\nu\left(\xi_0\right) \tag{28}$$

where ξ_0 is given by (24). By solving the quantization condition (28) for ν imposing that the solutions of (26) vanish for $\xi \to +\infty$, one obtains the possible energy levels by inserting those allowed values of ν in (24):

$$E=-mc^2\cos(\theta)\pm\sqrt{m^2c^4\cos(2\theta)+2\hbar c a_s(\nu+1)\sin^3(\theta)}$$

The numerical computation of (28) is substantially simpler when $D_{\nu+1}(\xi_0)$ is written in terms of $D'_\nu(\xi_0)$:

$$D'_\nu(\xi_0)=\left[\frac{\xi_0}{2}\mp\sqrt{\nu+1}\right]D_\nu(\xi_0) \tag{29}$$

Because the normalization of the spinor is not important for the calculation of the spectrum, one can arbitrarily choose $D_\nu(\xi_0)=1$. By using a fourth-fifth order Runge-Kutta method [16] an infinite sequence of allowed values of ν are found corresponding to each sign in (29). The lowest states are given in Table II for $\xi_0=1$.

It is worthwhile to note that the first class of solutions ($|V_s|=|V_t|$) is independent of the sign of a_s whereas the second class of solutions ($|V_s|>|V_t|$)depends on $|a_s\sin(\theta)|$. This observation permit us to conclude that even a "repulsive" potential can be a confining potential. This peculiar effect is because the scalar potential behaves like an x-dependent rest mass [14]. Nevertheless, only potentials with $a_s>0$ in the mixing $a_t=a_s\cos(\theta)$ are confining potentials in the nonrelativistic approximation and the case $V_t=-V_s$ reduces, for any sign of a_s, to the case of a free fermion. It is well known that a confining potential in the nonrelativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It surprising that relativistic confining potentials may result in nonconfinement in the nonrelativistic approach. This last phenomenon is a consequence of the fact that scalar and vector potentials couples differently in the Dirac equation whereas there is no such distinction between scalar and vector potentials in the Schrödinger equation. Therefore the results exposed in this paper, beyond to present a generalization of previous results, might be of relevance to the quarkonium phenomenology in a four-dimensional space-time.

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Table 1: The first roots of the Airy function and its first derivative

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$Ai'(-\|\beta_{\pm}\|)=0$	$Ai(-\|\beta_{\pm}\|)=0$
1.019	2.338
3.248	4.088
4.820	5.521

Table 2: The lowest solutions of Eq. (28) for $\xi_0 = 1$.

ν for the minus sign	ν for the plus sign
1.580×10^{-4}	1.525
2.681	3.915
5.038	6.210

References

[1] S. Coleman, R. Jackiw and L. Susskind, Ann. Phys. (N.Y.) **93**, 267 (1975).

[2] S. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976).

[3] J. Fröhlich and E. Seiler, Helv. Phys. Acta **49**, 889 (1976).

[4] A. Z. Capri and R. Ferrari, Can. J. Phys. **63**, 1029 (1985).

[5] H. Galić, Am. J. Phys. **56**, 312 (1988).

[6] G.´t Hooft, Nucl. Phys. B **75**, 461 (1974).

[7] J. Kogut and L. Susskind, Phys. Rev. D **9**, 3501 (1974).

[8] R. S. Bhalerao and B. Ram, Am. J. Phys. **69**, 817 (2001).

[9] A. S. de Castro, Am. J. Phys. **70**, 450 (2002).

[10] R. M. Cavalcanti, Am. J. Phys. **70**, 451 (2002).

[11] J. R. Hiller, Am. J. Phys. **70**, 522 (2002).

[12] W. Lucha, F. F. Schöberl and D. Gromes, Phys. Rep. **200**, 127 (1991) and references therein.

[13] A. N. Vaidya and R. L. Rodrigues, Phys. Rev. Lett. **89**, 068901 (2002).

[14] B. Thaller, *The Dirac Equation* (Springer-Verlag, Berlin, 1992).

[15] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, Toronto, 1965).

[16] G. E. Forsythe, M. A. Malcolm and C. B. Moler, *Computer Methods for Mathematical Computation* (Prentice-Hall, New Jersey, 1977).

[17] M. G. Salvadori, *Numerical Methods in Engineering* (Prentice-Hall, New York, 1952).

[18] I. S. Berezin and N. P. Zhidkov, *Computing Methods* vol II (Pergamon, Oxford, 1965).